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METRICS ON SEMISTABLE AND NUMERICALLY EFFECTIVE HIGGS BUNDLES

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Abstract. We provide notions of numerical effectiveness and numerical flatness for Higgs vector bundles on compact Kähler manifolds in terms of fibre metrics. We prove several properties of bundles satisfying such conditions and in particular we show that numerically flat Higgs bundles have vanishing Chern classes, and that they admit filtrations whose quotients are stable flat Higgs bundles. We compare these definitions with those previously given in the case of projective varieties. Finally we study the relations between numerically effectiveness and semistability, establishing semistability criteria for Higgs bundles on projective manifolds of any dimension.

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1. Introduction

Numerical effectiveness is a natural generalization of the notion of ampleness: a line bundle L on a projective variety X is said to be numerical effective (nef) if $c_1(L) \cdot [C] \geq 0$ for every irreducible curve C in X. A vector bundle E is said to be nef if the hyperplane bundle $\mathcal{O}_{\mathbb{P}E}(1)$ on the projectivized bundle $\mathbb{P}E$ is nef. A definition of numerical effectiveness can also be given in the case of Hermitian or Kähler manifolds in terms of (possibly singular) Hermitian fibre metrics.

The classification of the complex projective varieties or compact Kähler manifolds whose tangent bundle is numerically effective [9, 6] yields a far-reaching generalization of the Hartshorne-Frankel-Mori theorem, the latter stating that the complex projective n-space is the only projective n-variety whose tangent bundle is ample. Manifolds whose cotangent bundle is nef were studied by Kratz [19].

The concept of nefness may also be used to provide a characterization of semistable bundles. Several criteria of this type have appeared recently [5, 2, 3], all generalizing Miyaoka's result according to which a vector bundle E on a smooth projective curve X is semistable if and only if the numerical class $\lambda = c_1(\mathcal{O}_{\mathbb{P}E}(1)) - \frac{1}{r}\pi^*(c_1(E))$ is nef, where $\pi : \mathbb{P}E \to X$ is the projection. Also results by Gieseker [14], generalized in [5, 4], relate the notions of nefness and semistability.

A notion of nefness for Higgs bundles was introduced [4] in the case of projective varieties, and the basic properties of such bundles were there studied. In particular, it was proved there that numerically flat Higgs bundles (i.e., Higgs bundles that are numerically flat together with their duals) have vanishing Chern classes. One could stress here a similarity with the Bogomolov inequality: Higgs bundles that are semistable as Higgs bundles but not as ordinary bundles nevertheless satisfy Bogomolov's inequality.

The main purpose of this paper is to extend these constructions to the case when the base variety is a compact Kähler manifold. The guiding idea here is to give a definition of nefness in terms of fibre metrics on the bundles, following Demailly-Peternell-Schneider and de Cataldo [9, 7]. One takes the Higgs structure into account by replacing the Chern connection associated with the fibre metrics by a connection (introduced by Hitchin [17] and later extensively used by Simpson [23]) whose definition involves the Higgs field.

We now describe the contents of this paper. In Section 2, the Hitchin-Kobayashi correspondence for Higgs bundles is extended to a relation between semistability and the

existence of approximate Hermitian-Yang-Mills structures. This will be used in the ensuing sections for our study of numerically effectiveness of Higgs bundles. We also give Gauss-Codazzi equations associated with extensions of Higgs bundles.

In Section 3 we give our definition of numerically effectiveness and prove several related basic properties. The main result of this section is a characterization of numerically flat Higgs bundles on Kähler manifolds as those Higgs bundles which admit a filtration whose quotients are flat stable Higgs bundles (flat in a sense to be specified later). This implies that the Chern classes of numerically flat Higgs bundles vanish. We also establish some relations between semistability and numerical effectiveness.

In Section 4 we compare these results with our previous work in the case of projective varieties. This will also allow us to complete some of the results given in [4]. In particular, we provide new criteria for the characterization (in terms of the numerical effectiveness of certain associated Higgs bundles) of those Higgs bundles on projective manifolds that are semistable and satisfy the condition $\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = 0$.

We give now the basic definitions concerning Higgs bundles. Let X be an n-dimensional compact Kähler manifold, with Kähler form ω. Given a coherent sheaf F on X, we denote by $\deg(F)$ its *degree*

$$\deg(F) = \int_X c_1(F) \cdot \omega^{n-1}$$

and if $r = \mathrm{rk}(F) > 0$ we introduce its *slope*

$$\mu(F) = \frac{\deg(F)}{r}.$$

Definition 1.1. *A Higgs sheaf $\mathfrak{E}$ on X is a coherent sheaf E on X endowed with a morphism $\phi\colon E \to E \otimes \Omega_X$ of $\mathcal{O}_X$-modules such that the morphism $\phi \wedge \phi : E \to E \otimes \Omega^2_X$ vanishes, where Ω_X is the cotangent sheaf to X. A Higgs subsheaf F of a Higgs sheaf $\mathfrak{E} = (E, \phi)$ is a subsheaf of E such that $\phi(F) \subset F \otimes \Omega_X$. A Higgs bundle is a Higgs sheaf $\mathfrak{E}$ such that E is a locally-free $\mathcal{O}_X$-module.*

There exists a stability condition for Higgs sheaves, analogous to that for ordinary sheaves, which makes reference only to ϕ-invariant subsheaves.

Definition 1.2. *A Higgs sheaf $\mathfrak{E} = (E, \phi)$ on X is semistable (resp. stable) if E is torsion-free, and $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$) for every proper nontrivial Higgs subsheaf F of $\mathfrak{E}$.*

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2. Metrics and connections on semistable Higgs bundles

This section mostly deals with an application to Higgs bundles of the beautiful ideas underlying the so-called Hitchin-Kobayashi correspondence. Kobayashi and Lübke [18, 20] proved that the sheaf of sections of a holomorphic Hermitian (ordinary) vector bundle satisfying the Hermitian-Yang-Mills condition is polystable (i.e., it is a direct sum of stable sheaves having the same slope) A converse result was proved first by Donaldson in the projective case [10, 11], and then by Uhlenbeck and Yau in the compact Kähler case [25]; they showed that a stable bundle admits a (unique up to homotheties) Hermitian metric which satisfies the Hermitian-Yang-Mills condition. One can also show that if Hermitian bundle satisfies the Hermitian-Yang-Mills condition in an approximate sense, then it is semistable, while the converse may be proved if X is projective [18],

Later Simpson [22, 23] proved a Hitchin-Kobayashi correspondence for Higgs bundles. Given a Higgs bundle equipped with an Hermitian metric, one defines a natural connection (that we call the *Hitchin-Simpson connection*); when this satisfies the Hermitian-Yang-Mills condition then the Higgs bundle is polystable, and *vice versa* (this connection was originally introduced by Hitchin in [17]). Our main aim in this section is to show that whenever an Hermitian Higgs bundle satisfies an approximate Hermitian-Yang-Mills condition, then it is semistable. To this end we shall need to prove a vanishing result.

2.1. **Main definitions.** Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over a complex manifold X equipped with an Hermitian fibre metric h. Then there is on E a unique connection $D_{(E,h)}$ which is compatible with both the metric h and the holomorphic structure of E. This is often called the *Chern connection* of the Hermitian bundle (E, h).

Now let $\bar{\phi}$ be the adjoint of the morphism ϕ with respect to the metric h, i.e., the morphism $\bar{\phi} : E \to \Omega^{0,1} \otimes E$ such that

$$h(s, \phi(t)) = h(\bar{\phi}(s), t)$$

for all sections s, t of E. It is easy to check that the operator

$$\mathcal{D}_{(\mathfrak{E},h)} = D_{(E,h)} + \phi + \bar{\phi} \tag{1}$$

defines a connection on the bundle E. One should notice that this connection is neither compatible with the holomorphic structure of E, nor with the Hermitian metric h.

Definition 2.1. *The connection* (1) *is called the* Hitchin-Simpson connection *of the Hermitian Higgs bundle* $(\mathfrak{E}, h)$. *Its curvature will be denoted by* $\mathcal{R}_{(\mathfrak{E},h)} = \mathcal{D}_{(\mathfrak{E},h)} \circ \mathcal{D}_{(\mathfrak{E},h)}$. *If this curvature vanishes, we say that* $(\mathfrak{E}, h)$ *is* Hermitian flat.

If $\mathfrak{E}$ is a Higgs line bundle the notion of Hermitian flatness coincides with the usual one.

Let us now assume that X has a Kähler metric, with Kähler form ω. We shall denote by $\mathcal{K}_{(\mathfrak{E},h)} \in \operatorname{End}(E)$ the mean curvature of the Hitchin-Simpson connection. This is defined as usual: if we consider wedging by the Kähler 2-form as a morphism $\mathcal{A}^p \to \mathcal{A}^{p+2}$ (where $\mathcal{A}^p$ is the sheaf of $\mathbb{C}$-valued smooth p-forms on X), and denote by $\Lambda : \mathcal{A}^p \to \mathcal{A}^{p-2}$ its adjoint, then $\mathcal{K}_{(\mathfrak{E},h)} = i\Lambda\mathcal{R}_{(\mathfrak{E},h)}$. Locally, if we write

$$\mathcal{R}_{(\mathfrak{E},h)} = \tfrac{1}{2} \sum_{i,j,\alpha,\beta} (\mathcal{R}_{(\mathfrak{E},h)})^i{}_{j\alpha\beta}\, e_i \otimes (e^*)^j \otimes dx^\alpha \wedge dx^\beta$$

in terms of a local basis $\{e_i\}$ of sections of E, and local real coordinates $\{x^1, \ldots, x^{2n}\}$ on X, we have

$$(\mathcal{K}_{(\mathfrak{E},h)})^l{}_j = -i \sum_{\alpha,\beta} (\omega^{-1})^{\alpha\beta}\, (\mathcal{R}_{(\mathfrak{E},h)})^l{}_{j\alpha\beta}\,.$$

We shall regard the mean curvature as a bilinear form on sections of E by letting

$$\mathcal{K}_{(\mathfrak{E},h)}(s,t) = h(\mathcal{K}_{(\mathfrak{E},h)}(s), t)\,.$$

Let us recall, for comparison and further use, the form that the Hitchin-Kobayashi correspondence acquires for Higgs bundles [23, Thm. 1].

Theorem 2.2. *A Higgs vector bundle* $\mathfrak{E} = (E, \phi)$ *over a compact Kähler manifold is polystable if and only if it admits an Hermitian metric* h *such that the curvature of the Hitchin-Simpson connection of* $(\mathfrak{E}, h)$ *satisfies the Hermitian-Yang-Mills condition*

$$\mathcal{K}_{(\mathfrak{E},h)} = c \cdot \mathrm{Id}_E$$

for some constant real number c.

The constant c is related to the slope of E:

$$c = \frac{2n\pi}{n!\,\mathrm{vol}(X)}\mu(E) \tag{2}$$

where $n = \dim(X)$ and

$$\mathrm{vol}(X) = \frac{1}{n!}\int_X \omega^n\,.$$

As in the case of ordinary bundles, the semistability of a Higgs bundle may be related to the existence of an approximate Hermitian-Yang-Mills structure, which is introduced by replacing the Chern connection by the Hitchin-Simpson connection. Before doing that, we need to write the Gauss-Codazzi equations associated with an extension of Hermitian Higgs bundles.

2.2. **Metrics on Higgs subbundles and quotient bundles.** Given a rank r Higgs bundle $\mathfrak{E} = (E, \phi)$, let $\mathfrak{S} = (S, \phi_{|S})$ be a rank p Higgs subbundle (thus, $\phi_{|S}(S) \subset S \otimes \Omega^1$). The quotient bundle Q has an induced Higgs field ϕ_Q, so that one has a quotient Higgs bundle $\mathfrak{Q} = (Q, \phi_Q)$. If h is an Hermitian metric on $\mathfrak{E}$ the orthogonal complement $S^\perp$ is isomorphic to Q in the C^∞ category, including the extra structure as Higgs bundles, so that one gets a C^∞ decomposition of $\mathfrak{E}$ into Higgs subbundles

$$\mathfrak{E} \simeq \mathfrak{S} \oplus \mathfrak{S}^\perp\,.$$

Note that this endows the bundle Q with an Hermitian metric h_Q.

Let $e_1, \cdots, e_p, e_{p+1}, \cdots, e_r$ be a local C^∞ unitary frame field for E, such that the first p sections form a local unitary frame for S, and the last $r-p$ ones yield a local unitary frame for $\mathfrak{S}^\perp \simeq \mathfrak{Q}$. We shall use the following notation for the indices ranging over the sections of the various bundles:

$$1 \le a, b, c \le p, \qquad p+1 \le \lambda, \mu, \nu \le r, \qquad 1 \le i, j, k \le r.$$

We introduce local connection 1-forms by letting

$$\begin{aligned}
\mathcal{D}_{(\mathfrak{E},h)}(e_a) &= \sum_{b=1}^{p} \omega_a^b \otimes e_b + \sum_{\lambda=p+1}^{r} \omega_a^\lambda \otimes e_\lambda \\
\mathcal{D}_{(\mathfrak{E},h)}(e_\lambda) &= \sum_{a=1}^{p} \omega_\lambda^a \otimes e_a + \sum_{\mu=p+1}^{r} \omega_\lambda^\mu \otimes e_\mu\,.
\end{aligned}$$

Let us furthermore set

$$A(e_a) = \sum_{\lambda=p+1}^{r} \omega_a^\lambda \otimes e_\lambda \qquad \text{and} \qquad B(e_\lambda) = \sum_{a=1}^{p} \omega_\lambda^a \otimes e_a \,. \tag{3}$$

Proposition 2.3. (i) *The 1-forms ω_a^b are local connection 1-forms for the Hitchin-Simpson connection $\mathcal{D}_{(\mathfrak{S},h_S)}$ of $(\mathfrak{S}, h_S)$;*

(ii) *the map defined by A is a $(1,0)$-form with values in $Hom(S, S^\perp)$;*

(iii) *the map B is a $(0,1)$-form with values in $Hom(S^\perp, S)$.*

Proof. The first statement follows from the fact that $\mathfrak{S}$ is a Higgs subbundle of $\mathfrak{E}$. So we have $\mathcal{D}_{(\mathfrak{E},h)}(e_a) = \mathcal{D}_{(\mathfrak{S},h_S)}(e_a) + A(e_a)$. Since both connections satisfy the Leibniz rule, A is f-linear. The same is true for B. Moreover, $\mathcal{D}_{(\mathfrak{E},h)}$ and $\mathcal{D}_{(\mathfrak{S},h_S)}$ have the same $(0,1)$ part, namely, $\bar\partial + \bar\phi$, so that A is of type (1,0). Analogously, B is of type $(0,1)$. □

We want now to relate the curvatures of the Hitchin-Simpson connections on the Higgs bundles $\mathfrak{S}$ and $\mathfrak{E}$ (namely, we want to write the relevant equations of the Gauss-Codazzi type). Since (for simplicity we omit here the tensor product symbols and the ranges of summation over the indices)

$$\begin{aligned} D^2_{(\mathfrak{E},h)}(e_a) &= D_{(\mathfrak{E},h)}[\,\Sigma_b\, \omega_a^b\, e_b + \Sigma_\lambda\, \omega_a^\lambda\, e_\lambda] = \\ &= \Sigma_b[d\omega_a^b\, e_b - \omega_a^b \wedge (\Sigma_c\, \omega_b^c\, e_c + \Sigma_\mu\, \omega_b^\mu)\, e_\mu] + \\ &+ \Sigma_\lambda[d\omega_a^\lambda\, e_\lambda - \omega_a^\lambda \wedge (\Sigma_c\, \omega_\lambda^c\, e_c + \Sigma_\mu\, \omega_\lambda^\mu\, e_\mu)], \end{aligned}$$

one has

$$\mathcal{R}_{(\mathfrak{E},h)}(e_a) = \mathcal{R}_{(\mathfrak{S},h)}(e_a) + (B \wedge A)(e_a) + (\mathcal{D}_{(\mathfrak{S},h)} A)(e_a), \tag{4}$$

$$\mathcal{R}_{(\mathfrak{E},h)}(e_\lambda) = \mathcal{R}_{(\mathfrak{Q},h_Q)}(e_\lambda) + (A \wedge B)(e_\lambda) + (\mathcal{D}_{(\mathfrak{Q},h)} B)(e_\lambda). \tag{5}$$

One may note that the forms A and B are the same as in the ordinary case, i.e., they do not carry contributions from the Higgs fields.

2.3. **Approximate Hermitian-Yang-Mills structure and semistability.** For later use, we extend to the Higgs case the relation between the existence of approximate Hermitian-Yang-Mills structure and semistability. Given an Hermitian vector bundle (E, h), we introduce a norm on the space of Hermitian endomorphisms ψ of (E, h) by letting

$$|\psi| = \max_X \sqrt{\operatorname{tr}(\psi^2)}\,.$$

Definition 2.4. *We say that a Higgs bundle* $\mathfrak{E} = (E, \phi)$ *has an* approximate Hermitian-Yang-Mills structure *if for every positive real number* ξ *there is an Hermitian metric* h_ξ *on* E *such that*

$$|\mathcal{K}_{(\mathfrak{E},h)} - c \cdot \mathrm{Id}_E\,| < \xi\,. \tag{6}$$

The constant c is again given by equation (2).

We want to prove the following result (this is proved in the case of vector bundles in [18, VI.10.13]).

Theorem 2.5. *A Higgs bundle* $\mathfrak{E} = (E, \phi)$ *on a compact Kähler manifold admitting an approximate Hermitian-Yang-Mills structure is semistable.*

We shall need as usual a vanishing result.

Definition 2.6. *A section* s *of a Higgs bundle* $\mathfrak{E} = (E, \phi)$ *is said to be* ϕ-invariant *if it is an eigenvector of* ϕ*, that is, if there is a holomorphic 1-form* λ *on* X *such that* $\phi(s) = \lambda \otimes s$.

Proposition 2.7. *Assume that a Higgs bundle* $\mathfrak{E} = (E, \phi)$ *has an Hermitian metric* h *such that the mean curvature* $\mathcal{K}_{(\mathfrak{E},h)}$ *of the Hitchin-Simpson connection defines a seminegative definite form. Then* $D_{(E,h)}(s) = 0$ *and* $\mathcal{K}_{(\mathfrak{E},h)}(s, s) = 0$ *for every* ϕ*-invariant section* s *of* E.

Proof. We start by writing the relation between the curvatures of the Chern and Hitchin-Simpson connections for $(\mathfrak{E}, h)$. One has

$$\mathcal{R}_{(\mathfrak{E},h)} = R_{(E,h)} + D'_{(E,h)}(\phi) + D''_{(E,h)}(\bar\phi) + [\phi, \bar\phi] \tag{7}$$

where we have split the Chern connection $D_{(E,h)} = D'_{(E,h)} + D''_{(E,h)}$ into its (1,0) and (0,1) parts, and $[\phi, \bar\phi] = \phi \circ \bar\phi + \bar\phi \circ \phi$.

We now compute the Hitchin-Simpson curvature on a ϕ-invariant section s of E. Writing $\phi(s) = \lambda \otimes s$, one has

$$[\phi, \bar\phi](s) = 0\,, \qquad D'_{(E,h)}(\phi)(s) = \partial\lambda \otimes s\,, \qquad D''_{(E,h)}(\bar\phi)(s) = \bar\partial\bar\lambda \otimes s\,.$$

Thus we have

$$\mathcal{R}_{(\mathfrak{E},h)}(s) = R_{(E,h)}(s) + d(\lambda + \bar\lambda) \otimes s\,.$$

Still considering a ϕ-invariant section s, one has from here and from the Weitzenböck formula [1] the identity

$$\partial\bar\partial h(s, s)) = h\big(D'_{(E,h)}(s), D'_{(E,h)}(s)\big) - h\big(\mathcal{R}_{(\mathfrak{E},h)}(s), s\big) + h(s, s)\, d(\lambda + \bar\lambda)\,.$$

Let us set $f = h(s,s)$ and $L(f) = \Lambda(\partial\bar{\partial} f)$. By applying the operator Λ to the previous equation, we obtain, due to the hypotheses of this Proposition (and to the fact that the 2-form $d(\lambda + \bar{\lambda})$ has no (1,1) part)

$$L(f) = \|D'_{(E,h)}(s)\|^2 - \mathcal{K}_{(\mathfrak{E},h)}(s,s) \geq 0$$

where $\|D'_{(E,h)}(s)\|^2$ is the scalar product of $D'_{(E,h)}(s)$ with itself using the fibre metric h and the Kähler metric on X. By Hopf's maximum principle (see e.g. [18]) this implies $L(f) = 0$, which in turn implies $D'_{(E,h)}(s) = 0$ and $\mathcal{K}_{(\mathfrak{E},h)}(s,s) = 0$. Since s is holomorphic, and the Chern connection is compatible with the holomorphic structure of E, we also have $D_{(E,h)}(s) = 0$. □

Corollary 2.8. *Let $(\mathfrak{E}, h)$ be an Hermitian Higgs bundle. If the mean curvature $\mathcal{K}_{(\mathfrak{E},h)}$ of the Hitchin-Simpson connection is seminegative definite everywhere, and negative definite at some point of X, then E has no nonzero ϕ-invariant sections.*

Proof. If s is a nonzero ϕ-invariant section of E, then it never vanishes on X since $D_{(E,h)}(s) = 0$ by Proposition 2.7. By the same Proposition $\mathcal{K}_{(\mathfrak{E},h)}(s,s) = 0$, and this contradicts the fact that $\mathcal{K}_{(\mathfrak{E},h)}$ is negative at some point. □

Corollary 2.9. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over X which admits an approximate Hermitian-Yang-Mills structure. If $\deg(E) < 0$ then E has no nonzero ϕ-invariant sections.*

Proof. Since $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure, for every $\xi > 0$ there exists an Hermitian metric h_ξ on E such that

$$-\xi \cdot h_\xi < \mathcal{K}_{(\mathfrak{E},h_\xi)} - c \cdot h_\xi < \xi \cdot h_\xi$$

with $c < 0$. Then for ξ small enough $\mathcal{K}_{(\mathfrak{E},h_\xi)}$ is negative definite, and the result follows from the previous corollary. □

Proof of Theorem 2.5. Assume that $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure and let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E}$, with $\mathrm{rk}(F) = p$. Let $\mathfrak{G}$ be the Higgs bundle (G, ψ), where $G = \wedge^p E \otimes \det(F)^{-1}$, and ψ is the Higgs field naturally induced on G by the Higgs fields of $\mathfrak{E}$ and $\mathfrak{F}$. The inclusion $\mathfrak{F} \hookrightarrow \mathfrak{E}$ induces a morphism $\det(\mathfrak{F}) \to \wedge^p \mathfrak{E}$, and, tensoring by $\det(\mathfrak{F})^{-1}$, we obtain a ψ-invariant section of G. Now, it is not difficult to check that the Higgs bundle $\mathfrak{G}$ admits an approximate Hermitian-Yang-Mills structure, with constant

$$c_G = \frac{2np\pi}{n!\,\mathrm{vol}(X)}(\mu(E) - \mu(F))\,.$$

By Corollary 2.9 we have $c_G \geq 0$, so that $\mathfrak{E}$ is semistable.

3. Metric characterization of numerical effectiveness for Higgs bundles

The usual definitions of numerically effective line and vector bundle, which are given in terms of embedded curves, are not appropriate in the Kähler case since a Kähler manifold need not contain curves. The approaches by Demailly, Peternell and Schneider [9] and by de Cataldo [7] rely on a definition of numerical effectiveness in terms of fibre metrics. In particular, de Cataldo considers metrics on the bundle E, and this approach seems to be well suited to an extension to the case of Higgs bundles, again implementing the idea that the transition from the ordinary to the Higgs case is obtained by replacing the Chern connection with the Hitchin-Simpson connection.

3.1. **Numerically effective Higgs bundles.** De Cataldo's formalism rests on the following terminology. Let V, W be finite-dimensional complex vector spaces, and let θ_1, θ_2 be Hermitian forms on $V \otimes W$. Let t be any positive integer; one writes $\theta_1 \geq_t \theta_2$ if the Hermitian form $\theta_1 - \theta_2$ is semipositive definite on all tensors in $V \otimes W$ of rank $\rho \leq t$ (where the rank is that of the associated linear map $V^* \to W$). Of course the relevant range for t is $1 \leq t \leq N = \min(\dim V, \dim W)$.

If X is a compact Kähler manifold of dimension n and (E, h) is a rank r Hermitian vector bundle on X, equipped with a connection D, we may associate with the curvature R of D an Hermitian form $\widetilde{R}$ on $T_X \otimes E$, defined by

$$\widetilde{R}(u \otimes s, v \otimes t) = \frac{i}{2\pi} \langle h(R^{(1,1)}(s), t), u \otimes v \rangle \,. \tag{8}$$

where $R^{(1,1)}$ is the $(1,1)$ part of R. According to de Cataldo, the Hermitian bundle (E, h) is t-nef if for every $\xi > 0$ there exists an Hermitian metric h_ξ on E such that $\widetilde{R}_{(E,h_\xi)} \geq_t -\xi\omega \otimes h_\xi$.

Definition 3.1. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ on X is said to be*

(i) *t-H-semipositive if there is an Hermitian metric h on E such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h)} \geq_t 0$;*
(ii) *t-H-nef if for every $\xi > 0$ there exists an Hermitian metric h_ξ on E such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_t -\xi\omega \otimes h_\xi$;*
(iii) *t-H-nflat if both $\mathfrak{E}$ and $\mathfrak{E}^*$ are t-H-nef.*

Remark 3.2. (i) If $\mathfrak{E}$ is t-H-semipositive, then $\mathfrak{E}$ is t-H-nef. A t-H-semipositive (t-H-nef, respectively) Higgs bundle $\mathfrak{E}$ is t'-H-semipositive (t'-H-nef, respectively) for every t' such that $1 \leq t' \leq t$;

(ii) since the $(1,1)$ part of the Hitchin-Simpson curvature of a Higgs line bundle coincides with the Chern curvature, a Higgs line bundle is 1-H-nef if and only if it is 1-nef (as an ordinary bundle). More generally, since $\widetilde{\mathcal{R}}_{(\mathfrak{E},h)} \geq_t \widetilde{R}_{(E,h)}$, if E is t-nef then $\mathfrak{E} = (E, \phi)$ is t-H-nef for every choice of the Higgs field ϕ;

(iii) an Hermitian flat Higgs bundle is t-H-nflat for every t.

△

In the next Propositions we establish some basic properties of t-H-nef Higgs bundles on a compact Kähler manifold X.

Proposition 3.3. *Let $f : X \to Y$ be a holomorphic map, where X and Y are compact Kähler manifolds, and let $\mathfrak{E} = (E, \phi)$ be a t-H-nef Higgs vector bundle on Y; then $f^*\mathfrak{E} = (f^*E, f^*\phi)$ is a 1-H-nef Higgs bundle over X.*

Proof. This is proved as in [7, Proposition 3.2.1(1)]. □

Proposition 3.4. *Let $\mathfrak{E} = (E, \phi_E)$ and $\mathfrak{F} = (F, \phi_F)$ be Higgs bundles. If $\mathfrak{E}$ is t'-H-nef and $\mathfrak{F}$ is t''-H-nef, then $\mathfrak{E} \otimes \mathfrak{F} = (E \otimes F, \rho)$ is t-H-nef, where*

$$\begin{array}{rccc} \rho: & E \otimes F & \longrightarrow & E \otimes F \otimes \Omega^1 \\ & \rho(e \otimes f) & \mapsto & \phi_{\mathfrak{E}}(e) \otimes f + e \otimes \phi_{\mathfrak{F}}(f) \end{array} \tag{9}$$

and $t = \min(t', t'')$.

Proof. Since $\mathfrak{E} = (E, \phi_E)$ (analog., $\mathfrak{F} = (F, \phi_F)$) is t'-H-nef, for all $\xi > 0$ there exists a metric $h_{(\mathfrak{E},\xi/2)}$ over $\mathfrak{E}$ (analog. $h_{(\mathfrak{F},\xi/2)}$ over $\mathfrak{F}$) such that the Hitchin-Simpson curvature $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})}$ satisfies $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})} \geq_{t'} -\frac{\xi}{2}\omega \otimes h_{(\mathfrak{E},\xi/2)}$ (analogously, $\widetilde{\mathcal{R}}_{(\mathfrak{F},h_{(\mathfrak{F},\xi/2)})} \geq_{t''} -\frac{\xi}{2}\omega \otimes h_{(\mathfrak{F},\xi/2)}$). Considering on $\mathfrak{E} \otimes \mathfrak{F}$ the metrics $h_\xi = h_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})} \otimes h_{(\mathfrak{F},h_{(\mathfrak{F},\xi/2)})}$ we have

$$\widetilde{\mathcal{R}}_{(\mathfrak{E}\otimes\mathfrak{F},h_\xi)} = \widetilde{\mathcal{R}}_{(\mathfrak{E},h_{(\mathfrak{E},\xi/2)})} \otimes h_{(\mathfrak{F},\xi/2)} + h_{(\mathfrak{E},\xi/2)} \otimes \widetilde{\mathcal{R}}_{(\mathfrak{F},h_{(\mathfrak{F},\xi/2)})} \geq_t -\xi\omega \otimes h_\xi.$$

□

Analogously, one proves:

Proposition 3.5. *If $\mathfrak{E} = (E, \phi)$ is a t-H-nef Higgs bundle, then for all $p = 2, \ldots, r =$ $\mathrm{rk}(E)$ the p-th exterior power $\wedge^p\mathfrak{E} = (\wedge^p E, \wedge^p\phi)$ is a t-H-nef Higgs bundle, and for all m, the m-th symmetric power $S^m\mathfrak{E} = (S^m E, S^m\phi)$ is a t-H-nef Higgs bundle.*

Lemma 3.6. *Let $(\mathfrak{Q}, h_Q)$ be an Hermitian Higgs quotient of $(\mathfrak{E}, h)$. The respective Hitchin-Simpson curvatures verify the inequality $\widetilde{\mathcal{R}}_{(\mathfrak{Q},h_{\mathfrak{Q}})} \geq_1 \widetilde{\mathcal{R}}_{(\mathfrak{E},h)|\mathfrak{Q}}$.*

Proof. The Gauss-Codazzi equations (5) imply that the Hitchin-Simpson curvature of $\mathfrak{Q}$ is given by the restriction of the Hitchin-Simpson curvature of $\mathfrak{E}$ to $\mathfrak{Q}$ (if we embed $\mathfrak{Q}$ into $\mathfrak{E}$ by orthogonally splitting the latter) plus the semipositive term $A \wedge A^*$, where A^* is the Hermitian conjugate of A. □

Proposition 3.7. *A Higgs quotient $\mathfrak{Q}$ of a 1-H-nef Higgs bundle $\mathfrak{E} = (E, \phi)$ is 1-H-nef.*

Proof. Let $\xi > 0$ and h_ξ be an Hermitian metric on $\mathfrak{E}$ with $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$. We can endow $\mathfrak{Q}$ with the quotient metric $h_{(\mathfrak{Q},\xi)}$ and embed it into $\mathfrak{E}$ as a C^∞ Higgs subbundle. The claim follows from Lemma 3.6. □

Corollary 3.8. *If $0 \to \mathfrak{S} \to \mathfrak{E} \to \mathfrak{Q} \to 0$ is an exact sequence of Higgs bundles, with $\mathfrak{E}$ and $\det(\mathfrak{Q})^{-1}$ 1-H-nef, then $\mathfrak{S}$ is 1-H-nef.*

Proof. The proof is as in Proposition 1.15(iii) of [9]. Let $r = \mathrm{rk}(E)$ and $p = \mathrm{rk}(S)$. By taking the $(p-1)$-th exterior power of the morphism $\mathfrak{E}^* \to \mathfrak{S}^*$ obtained from the exact sequence in the statement, and using the isomorphism $\mathfrak{S} \simeq \wedge^{p-1}\mathfrak{S}^* \otimes \det(\mathfrak{E})$, we get a surjection $\wedge^{p-1}\mathfrak{E}^* \to \mathfrak{S} \otimes \det(\mathfrak{S})^{-1}$. Tensoring by $\det(\mathfrak{S}) \simeq \det(\mathfrak{E}) \otimes \det(\mathfrak{Q})^{-1}$ we eventually obtain a surjection $\wedge^{r-p+1}\mathfrak{E} \otimes \det(\mathfrak{Q})^{-1} \to \mathfrak{S}$. Propositions 3.4 and 3.7 now imply the claim. □

Proposition 3.9. *An extension of 1-H-nef Higgs bundles is 1-H-nef.*

Proof. Let us consider an extension of Higgs bundles

$$0 \to \mathfrak{F} \to \mathfrak{E} \to \mathfrak{G} \to 0$$

where $\mathfrak{F}$ and $\mathfrak{G}$ are 1-H-nef. Then for every $\xi > 0$ the latter bundles carry Hermitian metrics $h_{(\mathfrak{F},\xi)}$ and $h_{(\mathfrak{G},\xi)}$ such that

$$\widetilde{\mathcal{R}}_{(\mathfrak{F},h_{(\mathfrak{F},\xi/3)})} \geq_1 -\tfrac{\xi}{3}\omega \otimes h_{(\mathfrak{F},\xi/3)}, \qquad \widetilde{\mathcal{R}}_{(\mathfrak{G},h_{(\mathfrak{G},\xi/3)})} \geq_1 -\tfrac{\xi}{3}\omega \otimes h_{(\mathfrak{G},\xi/3)}.$$

Fixing a C^∞ isomorphism $\mathfrak{E} \simeq \mathfrak{F} \oplus \mathfrak{G}$, these metrics induce an Hermitian metric h_ξ on $\mathfrak{E}$. A simple calculation, which involves the second fundamental form of $\mathfrak{E}$, shows that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$, so that $\mathfrak{E}$ is 1-H-nef (details in the ordinary case are given in [7]). □

3.2. **Numerical effectiveness and semistability.** We study here the relations between the metric characterization of the numerical effectiveness of Higgs bundles and their semistability. The following result will be a useful tool.

Proposition 3.10. *Let $\mathfrak{E} = (E, \phi)$ be a 1-H-nef Higgs bundle, and $\mathfrak{E}^* = (E^*, \phi^*)$ the dual Higgs bundle. If s is a ϕ^*-invariant section of E^*, then s has no zeroes.*

Proof. We modify the proof of Proposition 1.16 in [9]. For a given $\xi > 0$, let h_ξ be the metric on E such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$. Let us define the closed $(1,1)$ current

$$T_\xi = \frac{i}{2\pi}\partial\bar{\partial}\log h^*_\xi(s,s);$$

a simple computation shows that it satisfies the inequality

$$T_\xi \geq -\frac{\widetilde{R}_{(E^*,h^*_\xi)}(s,s)}{h^*_\xi(s,s)},$$

where $\widetilde{R}_{(E^*,h^*_\xi)}(s,s)$ is regarded as a 2-form on X. Now, if s is ϕ^*-invariant, then $[\phi^*, \overline{\phi}^*](s) = 0$, so that $\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h^*_\xi)}(s,s) = \widetilde{R}_{(E^*,h^*_\xi)}(s,s)$. On the other hand, since $\mathfrak{E}$ is 1-H-nef, we have $-\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h^*_\xi)}(s,s) \geq -\xi\, h_\xi(s,s)\,\omega$. Thus, $T_\xi \geq -\xi\omega$.

Since $\partial\bar{\partial}\omega^{n-1} = 0$, we have

$$\int_X (T_\xi + \xi\omega) \wedge \omega^{n-1} = \xi \int_X \omega^n.$$

For ξ ranging in the interval $(0,1]$ the masses of the currents $T_\xi + \xi\omega$ are uniformly bounded from above, so that the sequence $\{T_\xi + \xi\omega\}$ contains a subsequence which, by weak compactness, converges weakly to zero. (For details on this technique see e.g. [8]). Therefore, the Lelong number of T_ξ at each point $x \in X$ (which coincides with the vanishing order of s at that point) is zero [24], which implies that s has no zeroes. □

Theorem 3.11. *A 1-H-nflat Higgs bundle $\mathfrak{E} = (E, \phi)$ is semistable.*

Proof. Since $\mathfrak{E}$ is 1-H-nef for every $\xi > 0$ it carries an Hermitian metric h_ξ such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\omega \otimes h_\xi$. As the mean curvature $\mathcal{K}_{(\mathfrak{E},h_\xi)}$ may be written in the form

$$\mathcal{K}_{(\mathfrak{E},h_\xi)}(s,s) = -2\pi \sum_{i=1}^{n} \widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}(e_i \otimes s, e_i \otimes s),$$

where the e_i's are a unitary frame field on X, one has

$$\mathcal{K}_{(\mathfrak{E},h_\xi)}(s,s) \leq 2\pi\, n\, \xi\, h_\xi(s,s).$$

On the other hand, since $\det(\mathfrak{E})^{-1}$ is 1-H-nef, the Higgs bundle $\mathfrak{E}^* \simeq \wedge^{r-1}\mathfrak{E} \otimes \det(\mathfrak{E})^{-1}$ is 1-H-nef with the dual metric h_ξ^*, so that $\mathcal{K}_{(\mathfrak{E}^*,h_\xi^*)} = -\mathcal{K}^t_{(\mathfrak{E},h_\xi)}$, and

$$\mathcal{K}_{(\mathfrak{E},h_\xi)}(s,s) \geq -2\pi\, n\, \xi\, h_\xi(s,s).$$

As $c_1(E) = 0$ because $\det(\mathfrak{E})$ is 1-nflat [9, Corollary 1.5], after rescaling ξ these equations imply $|\mathcal{K}_{(\mathfrak{E},h_\xi)}| \leq \xi$, so that $\mathfrak{E}$ is semistable by Theorem 2.5. □

3.3. **A characterization of 1-H-nflat Higgs bundles.** The next lemmas will be used in the proof of Theorem 3.16, which is one of the main results in this paper.

Lemma 3.12. *A 1-H-nef Higgs bundle $\mathfrak{E} = (E,\phi)$ such that $c_1(E) = 0$ is 1-H-nflat.*

Proof. This follows again from the fact that $\mathfrak{E}^* \simeq \wedge^{r-1}\mathfrak{E} \otimes \det(\mathfrak{E})^{-1}$ is an isomorphism of Higgs bundles. □

Lemma 3.13. *A 1-H-nef Higgs line bundle $\mathfrak{L}$ of zero degree is Hermitian flat.*

Proof. This is already contained in [9, Cor. 1.19], but for the reader's convenience we give here a proof. For every $\xi > 0$ one has on $\mathfrak{L}$ an Hermitian metric k_ξ satisfying the inequality

$$0 \leq \int_X \left(\tfrac{i}{2\pi}\mathcal{R}_{(\mathfrak{L},k_\xi)} + \xi\omega\right)\cdot\omega^{n-1} = \deg(L) + \xi \int_X \omega^n.$$

By the same argument as in the proof of Proposition 3.10, if $\deg(L) = 0$ by taking the limit $\xi \to 0$ one shows that $c_1(L) = 0$, so that $\mathfrak{L}$ is Hermitian flat. □

Lemma 3.14. *If the Higgs bundle $\mathfrak{E}$ is 1-Hnflat, and $\{h_\xi\}$ is a family of metrics which makes $\mathfrak{E}$ 1-H-nef, then the family of dual metrics $\{h^*_\xi\}$ makes $\mathfrak{E}^*$ 1-H-nef.*

Proof. The determinant line bundle $\det(E)$ is 1-nef with respect to the family of determinant metrics $\{\det h_\xi\}$. The dual line bundle $\det^{-1}(E)$ is 1-nef as well, and it is such with respect to a family $\{a(\xi)\,\det^{-1} h_\xi\}$, where the homothety factor $a(\xi)$ only depends on ξ [9, Cor. 1.5]. From the isomorphism $\mathfrak{E}^* \simeq \wedge^{r-1}\mathfrak{E} \otimes \det^{-1}(\mathfrak{E})$ (where $r = \mathrm{rk}(E)$) we see that $\mathfrak{E}^*$ is made 1-H-nef by the family of metrics $\{h'_\xi = a(\xi)\, h^*_\xi\}$, so that for every $\xi > 0$ the condition $\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h'_\xi)} \geq_1 -\xi\,\omega\otimes h'_\xi$ holds. But this implies $\widetilde{\mathcal{R}}_{(\mathfrak{E}^*,h^*_\xi)} \geq_1 -\xi\,\omega\otimes h^*_\xi$. □

Lemma 3.15. *A stable 1-H-nflat Higgs bundle $\mathfrak{E}$ is Hermitian flat (cf. Definition 2.1).*

Proof. As before, let us denote by $\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|^2$ the scalar product of the Hitchin-Simpson curvature with itself obtained by using the Hermitian metric of the bundle E and the Kähler form on X (thus, $\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|$ is a function on X). Note that in terms of a local

orthonormal frame $\{e_\alpha\}$ on X and a local orthonormal basis of sections $\{s_a\}$ of E we may write

$$\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|^2 = 4\pi^2 \sum_{\alpha,a} \left(\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}(e_\alpha \otimes s_a, e_\alpha \otimes s_a)\right)^2 .$$

Since $\mathfrak{E}$ is 1-H-nef, for every $\xi > 0$ there is an Hermitian metric h_ξ on E such that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)} \geq_1 -\xi\, \omega \otimes h_\xi$. Taking Lemma 3.14 into account, for every ξ we have the inequalities

$$\xi \geq \widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}(e_\alpha \otimes s_a, e_\alpha \otimes s_a) \geq -\xi\, .$$

So we have $\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\| \leq a_1\, \xi$ for some constant a_1. In the same way we have $\|\mathcal{K}_{(\mathfrak{E},h_\xi)}\| \leq a_2\, \xi$ for some constant a_2.

Assume that where $n = \dim X > 1$. Since $c_1(E) = 0$, we have the representation formula [18, Chap. IV.4]

$$\int_X c_2(E) \cdot \omega^{n-2} = \frac{1}{8\pi^2\, n(n-1)} \int_X (\|\mathcal{R}_{(\mathfrak{E},h_\xi)}\|^2 - \|\mathcal{K}_{(\mathfrak{E},h_\xi)}\|^2)\, \omega^n$$

The previous inequalities imply $\int_X c_2(E) \cdot \omega^{n-2} = 0$. For every value of n, may apply Theorem 1 and Proposition 3.4 in [22] to show that $\mathfrak{E}$ admits an Hermitian metric whose corresponding Hitchin-Simpson curvature vanishes. □

Theorem 3.16. *A Higgs bundle $\mathfrak{E}$ is 1-H-nflat if and only if it has a filtration in Higgs subbundles whose quotients are Hermitian flat Higgs bundles. As a consequence, all Chern classes of a 1-H-nflat Higgs bundle vanish.*

Proof. Assume that $\mathfrak{E}$ has such a filtration. Then any quotient of the filtration is 1-H-nflat, and the claim follows from Proposition 3.9.

To prove the converse, let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E}$ of rank p. We have an exact sequence of Higgs sheaves

$$0 \to \det(\mathfrak{F}) \to \wedge^p \mathfrak{E} \to \mathfrak{G} \to 0,$$

where $\mathfrak{G}$ is not necessarily locally-free. Since $\det(\mathfrak{E})$ is 1-H-nflat we have $c_1(E) = 0$. By Theorem 3.11 $\wedge^p\mathfrak{E}$ is semistable, so that $\deg(F) \leq 0$. Let h_ξ be a family of Hermitian metrics which makes $\mathfrak{E}$ a 1-H-nef Higgs bundle, and let h_ξ^p be the induced metrics on $\wedge^p\mathfrak{E}$. After rescaling the dual metrics $(h_\xi^p)^*$ we obtain a family of metrics which makes $\wedge^p\mathfrak{E}^*$ a 1-H-nef Higgs bundle (cf. Lemma 3.14). Let U be the open dense subset of X where $\mathfrak{G}$ is locally free; then the metrics $(h_\xi^p)^*$ induce on $\det(\mathfrak{F})^{-1}_{|U}$ metrics making it 1-H-nef. These metrics extend to the whole of X, since they are homothetic by a constant factor to the

duals of the metrics induced on $\det(\mathfrak{F})$ by the metrics on $\wedge^p\mathfrak{E}$. Thus, $\det(\mathfrak{F})^{-1}$ is 1-H-nef. If $\deg(F)=0$ by Lemma 3.13 $\det(\mathfrak{F})$ is Hermitian flat, so that $\wedge^p\mathfrak{E}\otimes\det(\mathfrak{F})^{-1}$ is 1-H-nflat. Then by Proposition 3.10 the morphism of Higgs bundles $\det(\mathfrak{F})\to\wedge^p\mathfrak{E}$ has no zeroes, so that $\mathfrak{G}$ is locally-free.

In view of Lemma 3.15 we may assume that $\mathfrak{E}$ is not stable. Let us then identify $\mathfrak{F}$ with a destabilizing Higgs subsheaf of minimal rank and zero degree. We need $\mathfrak{F}$ to be reflexive; we may achieve this by replacing $\mathfrak{F}$ with its double dual $\mathfrak{F}^{**}$. By Lemma 1.20 in [9], $\mathfrak{F}$ is locally-free and a Higgs subbundle of $\mathfrak{E}$. Now, $\mathfrak{F}^*$ is 1-H-nef because it is a Higgs quotient of $\mathfrak{E}^*$, while $\mathfrak{F}$ is 1-H-nef by Corollary 3.8, so that $\mathfrak{F}$ is 1-H-nflat. Since $\mathfrak{F}$ is stable by construction, by Lemma 3.15 it is Hermitian flat. The existence of the filtration follows by induction on the rank of $\mathfrak{E}$ since the quotient $\mathfrak{E}/\mathfrak{F}$ is locally-free and 1-H-nflat, hence we may apply to it the inductive hypothesis. □

3.4. **Projective curves.** We conclude this section by proving two results that hold when X is a smooth projective curve. The first Proposition generalizes results given in [14, 5, 4].

Proposition 3.17. *If a Higgs bundle $\mathfrak{E}$ on X is semistable and $\deg(E)\geq 0$, then $\mathfrak{E}$ is 1-H-nef.*

Proof. If $\mathfrak{E}$ is stable it admits an Hermitian-Yang-Mills metric h, so that $\widetilde{\mathcal{R}}_{(\mathfrak{E},h)}=c\,h$ with $c\geq 0$ (note that we essentialy identify $\widetilde{\mathcal{R}}_{(\mathfrak{E},h_\xi)}$ with the mean curvature since we are on a curve). Then $\mathfrak{E}$ is 1-H-nef.

If $\mathfrak{E}$ is properly semistable, we may filter it in such a way that the quotients of the filtration are stable Higgs bundles of nonnegative degree. By the previous argument, every quotient is 1-H-nef. One then concludes by Proposition 3.9. □

In [16] Hartshorne gives a characterization of ample vector bundles of rank 2 on a smooth projective curve. Our next result generalizes this to 1-H-nef Higgs bundles of any rank on a smooth projective curve. A similar statement might be easily obtained for the ample case. This also partly generalizes Theorem 3.3.1 in [7].

Proposition 3.18. *Let $\mathfrak{E}$ be a Higgs bundle of nonnegative degree over a smooth projective curve X, whose locally-free Higgs quotients are all 1-H-nef. Then $\mathfrak{E}$ is 1-H-nef.*

Proof. One may consider two cases:

1.- If $\mathfrak{E}$ is semistable then by Proposition 3.17 it is 1-H-nef.

2.- If $\mathfrak{E}$ is not semistable, then it has a semistable Higgs subbundle $\mathfrak{K}$ with $\mu(K) > \mu(E)$, so that one has a exact sequence of Higgs bundles $0 \to \mathfrak{K} \to \mathfrak{E} \to \mathfrak{Q} \to 0$. Since $\deg(E) \geq 0$, then $\deg(K) > 0$, and again by Proposition 3.17 we have that $\mathfrak{K}$ is 1-H-nef. Thus $\mathfrak{E}$ is an extension of 1-H-nef Higgs bundles, so that it is 1-H-nef by Proposition 3.9. □

Remark 3.19. It seems useful to state in a close way the relations between the conditions of 1-H-nefness and semistability in the case of Higgs bundles on curves. So, let X be a complex smooth projective curve, and $\mathfrak{E} = (E, \phi)$ a Higgs bundle on X. Let $d = \deg(E)$.

(i) If $\mathfrak{E}$ is semistable and $d > 0$, then $\mathfrak{E}$ is 1-H-nef.
(ii) If $\mathfrak{E}$ is semistable and $d < 0$, then $\mathfrak{E}^*$ is 1-H-nef.
(iii) if $\mathfrak{E}$ is semistable and $d = 0$, then $\mathfrak{E}$ is 1-H-nflat.

On the other hand, if $\mathfrak{E}$ is 1-H-nef and $d \neq 0$, then it need not be semistable (an example is provided by a direct sum of 1-H-nef Higgs bundles of different slopes). However, if $\mathfrak{E}$ is 1-H-nef and $d = 0$ (i.e., it is 1-H-nflat) we know it is semistable (Theorem 3.11; this also follows from Corollary 3.6 of [4] since, as we shall see in the next Section, on a curve the notions of H-nefness and 1-H-nefness coincide). △

4. The projective case

In our previous paper [4] we gave a definition of numerical flatness for Higgs bundles on smooth projective varieties. In this section we want to compare it with the definition we have given here in the case of Kähler manifolds. We start by briefly recapping the situation in the projective case.

4.1. **Grassmannians of Higgs quotients.** Our definition of H-nefness for Higgs bundles on projective varieties requires to consider Higgs bundles on singular spaces (the Higgs Grassmannians whose definition we are going to recall in this Section, see also [5, 4]). As we shall see, Higgs Grassmannians are locally defined as the zero locus of a set of holomorphic functions on the usual Grassmannian varieties, and are therefore schemes. For such spaces there is well-behaved theory of the de Rham complex [15], which is all one needs to define Higgs bundles. Let us in particular notice that Higgs bundles on schemes are well-behaved with respect to restrictions to closed subschemes. Indeed, if $\mathfrak{E} = (E, \phi)$ is a Higgs bundle on a scheme X, and $Y \hookrightarrow X$ is a closed immersion, due to the isomorphism

$$\begin{aligned} (E \otimes_{\mathcal{O}_X} \Omega_X)_{|Y} &= (E \otimes_{\mathcal{O}_X} \Omega_X) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \\ &\simeq (E \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \simeq E_{|Y} \otimes_{\mathcal{O}_Y} \Omega_{X|Y} , \end{aligned}$$

by composing the restriction $\phi_{|Y}\colon E_{|Y} \to E_{|Y} \otimes_{\mathcal{O}_Y} \Omega_{X|Y}$ with the projection $p\colon \Omega_{X|Y} \to \Omega_Y$ we obtain a Higgs bundle $\mathfrak{E}_{|Y} = (E_{|Y}, (1 \times p) \circ \phi_{|Y})$ on Y.

Moreover, the Chern classes we shall be using in that case may be regarded as those in the Fulton-MacPherson theory, cf. [13].

Thus, let X be a scheme over the complex numbers. Given a rank r vector bundle E on X, for every positive integer s less than r, let $\mathrm{Gr}_s(E)$ be the Grassmann bundle of s-planes in E, with projection $p_s : \mathrm{Gr}_s(E) \to X$. This is a parametrization of the rank s locally-free quotients of E. There is a universal exact sequence

$$0 \to S_{r-s,E} \xrightarrow{\psi} p_s^*(E) \xrightarrow{\eta} Q_{s,E} \to 0 \tag{10}$$

of vector bundles on $\mathrm{Gr}_s(E)$, with $S_{r-s,E}$ the universal rank $r-s$ subbundle and $Q_{s,E}$ the universal rank s quotient bundle [13].

Given a Higgs bundle $\mathfrak{E}$, we may construct closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ parametrizing rank s locally-free Higgs quotients. With reference to the exact sequence eq. (10), we define $\mathfrak{Gr}_s(\mathfrak{E})$ as the closed subscheme of $\mathrm{Gr}_s(E)$ where the composed morphism

$$(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi\colon S_{r-s,E} \to Q_{s,E} \otimes p_s^*(\Omega_X)$$

vanishes. We denote by ρ_s the projection $\mathfrak{Gr}_s(\mathfrak{E}) \to X$. The restriction of (10) to the scheme $\mathfrak{Gr}_s(\mathfrak{E})$ provides the exact sequence (of vector bundles, for the moment)

$$0 \to S_{r-s,\mathfrak{E}} \to \rho_s^*(\mathfrak{E}) \to Q_{s,\mathfrak{E}} \to 0\,. \tag{11}$$

The Higgs morphism ϕ of $\mathfrak{E}$ induces by pullback a Higgs morphism $\Phi\colon \rho_s^*(\mathfrak{E}) \to \rho_s^*(\mathfrak{E}) \otimes \Omega_{\mathfrak{Gr}_s(\mathfrak{E})}$ (note in particular that $\Phi \wedge \Phi = 0$). On the account of the condition $(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi = 0$ which holds true on $\mathfrak{Gr}_s(\mathfrak{E})$, the morphism Φ sends $S_{r-s,\mathfrak{E}}$ to $S_{r-s,\mathfrak{E}} \otimes \Omega_{\mathfrak{Gr}_s(\mathfrak{E})}$. Therefore, $S_{r-s,\mathfrak{E}}$ is a Higgs subbundle of $\rho_s^*(\mathfrak{E})$, and the quotient $Q_{s,\mathfrak{E}}$ inherits a structure of Higgs bundle as well. The sequence (11) is therefore an exact sequence of Higgs bundles.

By its very construction, the scheme $\mathfrak{Gr}_s(\mathfrak{E})$ and the quotient bundle $Q_{s,\mathfrak{E}}$ enjoy a universal property: for every morphism $f\colon Y \to X$ and every rank s Higgs quotient $\mathfrak{F}$ of $f^*(\mathfrak{E})$ there is a morphism $g\colon Y \to \mathfrak{Gr}_s(\mathfrak{E})$ such that $f = \rho_s \circ g$ and $\mathfrak{F} \simeq g^*(Q_{s,\mathfrak{E}})$ as Higgs bundles. Therefore, the scheme $\mathfrak{Gr}_s(\mathfrak{E})$ will be called the *Grassmannian of locally-free rank s Higgs quotients* of $\mathfrak{E}$, and $Q_{s,\mathfrak{E}}$ will be called the *rank s universal Higgs quotient vector bundle.*

It is useful to introduce the following numerical classes

$$\lambda_{s,\mathfrak{E}} = \left[c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1) - \frac{1}{r}\pi_s^*(c_1(E))\right] \in N^1(\mathbb{P}Q_{s,\mathfrak{E}}) \tag{12}$$

$$\theta_{s,\mathfrak{E}} = \left[c_1(Q_{s,\mathfrak{E}}) - \frac{s}{r}\,\rho_s^*(c_1(E))\right] \in N^1(\mathfrak{Gr}_s(\mathfrak{E})), \tag{13}$$

where, for every projective scheme Z, we denote by $N^1(Z)$ the vector space of $\mathbb{R}$-divisors modulo numerical equivalence:

$$N^1(Z) = \frac{\mathrm{Pic}(Z)}{num.\,eq.} \otimes \mathbb{R}.$$

4.2. **Comparison between the projective and Kählerian cases.** We recall from [4] the notion of H-nef Higgs bundle.

Definition 4.1. *A Higgs bundle $\mathfrak{E}$ of rank one is said to be Higgs-numerically effective (for short, H-nef) if it is numerically effective in the usual sense. If* $\operatorname{rk}\mathfrak{E} \geq 2$ *we require that:*

(i) *all bundles $Q_{s,\mathfrak{E}}$ are Higgs-nef;*
(ii) *the line bundle* $\det(E)$ *is nef.*

If both $\mathfrak{E}$ and $\mathfrak{E}^*$ are Higgs-numerically effective, $\mathfrak{E}$ is said to be Higgs-numerically flat (H-nflat).

If the Higgs field is zero (i.e., $\mathfrak{E}$ is an ordinary vector bundle) this definition reduces to the usual one.

Remark 4.2. Due to our iterative definition of H-nefness, a Higgs bundle $\mathfrak{E}$ is H-nef if and only if a finite number of line bundles L_i (each defined on a projective scheme Y_i for which a surjective morphism $Y_i \to X$ exists) are nef. For instance, if $\mathfrak{E}$ is a rank 3 Higgs bundle of X, one is requiring the usual nefness of the following line bundles:

- $\det(\mathfrak{E})$ on X
- $Q_{1,\mathfrak{E}}$ on $\mathfrak{Gr}_1(\mathfrak{E})$
- $\det(Q_{2,\mathfrak{E}})$ on $\mathfrak{Gr}_2(\mathfrak{E})$
- $Q_{1,Q_{2,\mathfrak{E}}}$ on $\mathfrak{Gr}_1(Q_{2,\mathfrak{E}})$.

$\triangle$

Proposition 4.3. *A 1-H-nef Higgs bundle $\mathfrak{E} = (E, \phi)$ is H-nef.*

Proof. We proceed by induction on the rank r of $\mathfrak{E}$. If $r = 1$ there is nothing to prove. If $r > 1$, for every $s = 1, \ldots, r-1$ let us consider the universal sequence (11) on the Higgs Grassmannian $\mathfrak{Gr}_s(\mathfrak{E})$. Since the Higgs Grassmannian is in general singular, we consider a resolution of singularities $\beta_s : B_s(\mathfrak{E}) \to \mathfrak{Gr}_s(\mathfrak{E})$, and pullback the universal sequence to $B_s(\mathfrak{E})$:

$$0 \to \beta_s^* S_{r-s,\mathfrak{E}} \to \gamma_s^* \mathfrak{E} \to \beta_s^* Q_{s,\mathfrak{E}} \to 0,$$

where $\gamma_s = \rho_s \circ \beta_s$. Since $\mathfrak{E}$ is 1-H-nef, the pullback $\gamma_s^*(\mathfrak{E})$ is 1-H-nef as well, and its Higgs quotient $\beta_s^* Q_{s,\mathfrak{E}}$ is 1-H-nef, hence H-nef by the inductive hypothesis.

We need to show that $Q_{s,\mathfrak{E}}$ is H-nef; in view of Remark 4.2, by base change this reduces to proving the following fact: if $f_i : Z_i \to Y_i$ are surjective morphisms of projective schemes, and L_i are line bundles on Y_i such that the pullbacks $f_i^* L_i$ are nef, then the line bundles L_i are nef. This follows from [12, Prop. 2.3]. □

Proposition 4.4. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ over a smooth projective curve X is 1-H-nef if and only if it is H-nef.*

Proof. We have just proved the necessary condition. We prove the sufficiency again by induction on the rank r of $\mathfrak{E}$. If $r = 1$ there is nothing to prove. If $r > 1$, note that since $\mathfrak{E}$ is H-nef, then $\deg(E) \geq 0$, and all its quotients $\mathfrak{Q}$ are H-nef. By the inductive hypothesis, all $\mathfrak{Q}$ are 1-H-nef; one concludes by Proposition 3.18. □

This strongly simplifies the proof of Theorem 3.3.1 of [7], which gives the same result in the case of ordinary bundles.

We may use these results to prove some properties of H-nef Higgs bundles in addition to those given in [4].

Lemma 4.5. *A Higgs bundle $\mathfrak{E} = (E, \phi)$ over a smooth projective variety X is H-nef if and only if $\mathfrak{E}_{|C} = (E_{|C}, \phi_{|C})$ is H-nef for all irreducible curves C in X.*

Proof. By Remark 4.2 the Higgs bundle $\mathfrak{E}$ is H-nef if and only if a finite number of line bundles L_i (each defined on a projective scheme Y_i for which a surjective morphism $Y_i \to X$ exists) are nef. The claim then follows. □

Remark 4.6. We should note that the fact that a Higgs bundle restricts to a semistable Higgs bundle on any embedded curve is not enough to ensure that it is H-nef. Consider for instance a a surface with Picard number 1, and let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle of

negative degree which is semistable after restriction to every curve (i.e., it is semistable and $\Delta(E) = 0$). The Higgs bundle $\mathfrak{E}$ cannot be H-nef since it has negative degree. △

Proposition 4.7. *Let $0 \to \mathfrak{F} \to \mathfrak{E} \to \mathfrak{G} \to 0$ be an exact sequence of Higgs bundles over a smooth projective variety X. If $\mathfrak{F}$ and $\mathfrak{G}$ are H-nef then $\mathfrak{E}$ is H-nef.*

Proof. In view of Lemma 4.5 we may assume that X is a curve. The result then follows from Propositions 3.9 and 4.4. □

In the same way, by using Lemma 4.5 one can prove that the tensor, exterior and symmetric products of H-nef Higgs bundles are H-nef, thus completing the results given in [4]. Moreover we have:

Proposition 4.8. *Let $\mathfrak{E}$ be a Higgs bundle. If $S^m(\mathfrak{E})$ is H-nef for some m, then $\mathfrak{E}$ is H-nef.*

Proof. Since a rank s Higgs quotient of $\mathfrak{E}$ yields a Higgs quotient of $S^m(\mathfrak{E})$ of rank

$$N_{(m,s)} = \binom{m+s-1}{s-1},$$

one has a morphism $g : \mathfrak{Gr}_s(\mathfrak{E}) \to \mathfrak{Gr}_{N_{(m,s)}}(S^m(\mathfrak{E}))$ such that $g^*(Q_{N_{(m,s)},S^m(\mathfrak{E})}) \simeq S^m(Q_{s,\mathfrak{E}})$. Since $S^m(\mathfrak{E})$ is H-nef, the symmetric product $S^m(Q_{s,\mathfrak{E}})$ is H-nef. The claim follows by induction on the rank of $\mathfrak{E}$. □

4.3. **Semistability criteria.** The notion of H-nefness may be used to provide a characterization of a special class of Higgs bundles, namely, the semistable Higgs bundes for which the cohomology class $\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2$ vanishes. Recently several similar criteria have appeared in the literature, which all generalize Miyaoka's criterion for the semistability of vector bundles on projective curves [21]. In [5] a criterion was given for characterizing semistable Higgs bundles with vanishing Δ class on complex projective manifolds of any dimension in terms of the nefness of a certain set of divisorial classes (it is interesting to note that Higgs bundles of this type have the property of being semistable after restriction to any curve in the base manifold). Analogous criteria have been formulated for principal bundles on complex projective manifolds [2], and more generally for principal bundles on compact Kähler manifolds [3].

We discuss now three equivalent conditions of this kind. One of these is stated in terms of the Higgs bundles $T_{s,\mathfrak{E}} = S^*_{r-s,\mathfrak{E}} \otimes Q_{s,\mathfrak{E}}$ on the Higgs Grassmannians $\mathfrak{Gr}_s(\mathfrak{E})$. For an ordinary vector bundle E, the bundle $T_{s,E}$ is the vertical tangent bundle to $p_s : \mathrm{Gr}_s(E) \to X$.

Theorem 4.9. *Let $\mathfrak{E} = (E, \phi)$ be a rank r Higgs bundle on a complex projective manifold X. The following three conditions are equivalent:*

(i) *The Higgs bundle $\mathfrak{F} = S^r(\mathfrak{E}) \otimes (\det \mathfrak{E})^{-1}$ is H-nflat;*
(ii) *$\mathfrak{E}$ is semistable and $\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = 0$;*
(iii) *the Higgs bundles $T_{s,\mathfrak{E}}$ are all H-nef.*

Proof. We first prove that (i) implies (ii). Since $\det(\mathfrak{F})$ is trivial, the dual Higgs bundle $\mathfrak{F}^*$ is H-nef as well, i.e., $\mathfrak{F}$ is H-nflat, hence semistable by Theorem 3.1 of [4]. Then, the Higgs bundle $\mathfrak{F} \otimes \mathfrak{F}^* \simeq S^r(\mathfrak{E}) \otimes S^r(\mathfrak{E}^*)$ is semistable. This implies that $\mathfrak{E}$ is semistable.

One also has that $S^r(\mathfrak{E}) \otimes S^r(\mathfrak{E}^*)$ is H-nflat so that its Chern classes vanish by Corollary 3.2 of [4]. But since

$$c_2(S^r(E) \otimes S^r(E^*)) = 4r(\operatorname{rk} S^r(E))^2 \Delta(E)$$

we conclude.

(ii) implies (i): we have that $\mathfrak{F}$ is semistable and

$$c_1(F) = 0, \qquad c_2(F) = 2r(\operatorname{rk} S^r(E))^2 \Delta(E) = 0.$$

By Theorem 2 of [23], $\mathfrak{F}$ has a filtration whose quotients are stable Higgs bundles with vanishing Chern classes. Proceeding as in the proof of Lemma 3.15, these quotients are shown to be Hermitian flat, hence they are H-nflat. Then $\mathfrak{F}$ is H-nflat as well.

We prove now that (i) implies (iii). If $\mathfrak{F}$ is H-nef, then the $\mathbb{Q}$-Higgs bundle $\mathfrak{E} \otimes (\det(\mathfrak{E}))^{-1/r}$ is H-nef by Proposition 4.8, so that the Higgs bundle $\mathfrak{E}^* \otimes (\det(\mathfrak{E}))^{1/r}$ is H-nef (since $c_1(\mathfrak{E} \otimes (\det(\mathfrak{E}))^{-1/r}) = 0$), and $Q_{s,\mathfrak{E}} \otimes \rho_s^*(\det(\mathfrak{E}))^{-1/r}$ is H-nef as well, since it is a universal quotient of $\mathfrak{E} \otimes (\det(\mathfrak{E}))^{-1/r}$. From the exact sequence

$$0 \to Q_{s,\mathfrak{E}} \otimes Q_{s,\mathfrak{E}}^* \to \rho_s^*(\mathfrak{E}^*) \otimes Q_{s,\mathfrak{E}} \to T_{s,\mathfrak{E}} \to 0,$$

one obtains the claim.

Finally, we prove that (iii) implies (ii). Note that the class $\theta_{s,\mathfrak{E}}$ defined in equation (13) equals $[c_1(T_{s,\mathfrak{E}})]$, so that if $T_{s,\mathfrak{E}}$ is H-nef, the class $\theta_{s,\mathfrak{E}}$ is nef. This holds true for every $s = 1, \ldots, r-1$. It was proved in [5] that this is equivalent to condition (ii) in the statement. □

Remark 4.10. In [5] it was proved that condition (ii) is fulfilled if and only if all classes $\lambda_{s,\mathfrak{E}}$ are nef. It may be interesting to check directly that the latter condition is equivalent to condition (i) in our Theorem 4.9.

If $\mathfrak{F} = S^r\mathfrak{E} \otimes \det(\mathfrak{E})^{-1}$ is H-nef, since $c_1(F) = 0$ it is also H-nflat, so that all its Higgs quotients are nef in the usual sense (see [4, Cor. 3.6]). Since the Higgs bundle $S^r Q_{s,\mathfrak{E}} \otimes \rho_s^*(\det(\mathfrak{E})^{-1})$ is a quotient of $\rho_s^*(\mathfrak{F})$, it is nef; moreover pulling it back to $\mathbb{P}Q_{s,\mathfrak{E}}$ it has a surjection onto $\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(r) \otimes \pi_s^*(\det(\mathfrak{E}))^{-1} \simeq \mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(r\lambda_{s,\mathfrak{E}})$, so that $\lambda_{s,\mathfrak{E}}$ is nef.

Conversely, if $\lambda_{s,\mathfrak{E}}$ is nef, the $\mathbb{Q}$-Higgs bundle $Q_{s,\mathfrak{E}} \otimes \rho_s^*(\det(\mathfrak{E})^{-1/r})$ is nef. Since this is true for every s, the $\mathbb{Q}$-Higgs bundle $\mathfrak{E} \otimes \det(\mathfrak{E})^{-1/r}$ is H-nef; by taking the r-th symmetric power we obtain that $\mathfrak{F}$ is H-nef. $\triangle$

Example 4.11. We give an example of an H-nef Higgs bundle which is not nef as an ordinary bundle. Let X be a projective surface of general type that saturates Miyaoka-Yau's inequality, i.e., $3c_2(X) = c_1(X)^2$ (surfaces of general type satisfying this condition are exactly those that are uniformized by the unit ball in $\mathbb{C}^2$ [22]). The Higgs bundle $\mathfrak{E}$ whose underlying vector bundle is $E = \Omega_X \oplus \mathcal{O}_X$ with the Higgs morphism $\phi(\omega, f) = (0, \omega)$ is semistable and satisfies $\Delta(E) = 0$, so that the Higgs bundle $\mathfrak{F} = S^3(\mathfrak{E}) \otimes (\det \mathfrak{E})^{-1}$ is 1-H-nef. On the other hand, the underlying vector bundle $F = S^3(\Omega_X \oplus \mathcal{O}_X) \otimes K_X^*$ contains K_X^* as a direct summand and therefore is not nef (note that we exclude that K_X is numerically flat). $\triangle$

From this Example we see that if X is a projective surface of general type for which the Miyaoka-Yau inequality holds strictly, i.e., $3c_2(X) > c_1(X)^2$, then there are curves C in X such that the restriction of the Simpson system $E = \Omega_X \oplus \mathcal{O}_X$ to C is not semistable. It would interesting to investigate what this tells us about the geometry of such curves.

It is interesting to note that the criterion expressed in Theorem 4.9 allows one to relate the semistability of a vector bundle E with the nefness of the tangent bundle to the total space of the Grassmann varieties of E.

Example 4.12. Let X be a complex projective manifold, and E a rank r vector bundle on X. Let us choose an integer s such that $0 < s < r$, and consider the exact sequence

$$0 \to T_{\mathrm{Gr}_s(E)/X} \to T_{\mathrm{Gr}_s(E)} \to p_s^*(T_X) \to 0\,.$$

We have the following results:

(i) If E is semistable and $\Delta(E) = 0$, and T_X is nef, then $T_{\mathrm{Gr}_s(E)}$ is nef. Since the tangent bundles to the fibres of the projection $T_{\mathrm{Gr}_s(E)} \to T_X$ are nef, the fact the E is semistable and has vanishing discriminant appear to be the conditions for these bundles to glue to a nef bundle on $\mathrm{Gr}_s(E)$.

(ii) If $T_{\mathrm{Gr}_s(E)}$ and K_X^{-1} are nef, then E is semistable and $\Delta(E) = 0$.

(iii) As a consequence, when T_X is numerically flat (e.g., X is an Abelian variety, or a hyperelliptic surface) then $T_{\mathrm{Gr}_s(E)}$ is nef if and only if E is semistable and $\Delta(E) = 0$.

$\triangle$

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